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Complex Potentials

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Using basic principles from complex analysis, we prove that complex potentials associated with locally sourceless, locally irrotational plane flows only have logarithmic singularities. Applications to flows around contours are given. © 1999 Academic Press

1. INTRODUCTION

In this note we shed some light on the singularities of complex potentials associated with locally sourceless and locally irrotational flows $w = u + iv$, defined on (possibly infinitely connected) domains G in the complex plane $\mathbb{C}$. For example, consider the two-dimensional velocity vector $\mathbf{w} = (u, v)^t$ of an ideal fluid on G. Without further notice we adopt the convention of identifying the vector $(x, y)^t$ with the complex number $z = x + iy$, hence $\mathbf{w} = (u, v)^t$ with $w = u + iv$. By a *flow* $w = u + iv$ on a

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domain G we just mean any function w whose real and imaginary part is continuously differentiable on G.

Fix a closed differentiable curve $\gamma \subset G$. Then the *flux* N across γ and the *circulation* Γ along γ are defined as

$$N = \oint_\gamma \mathbf{w} \cdot \mathbf{n}\, ds = \oint_\gamma u\, dy - v\, dx \quad \text{and} \quad \Gamma = \oint_\gamma \mathbf{w} \cdot \mathbf{t}\, ds = \oint_\gamma u\, dx + v\, dy.$$

A calculation gives

$$\Gamma + iN = \oint_\gamma \overline{w(z)}\, dz.$$

Recall that the flow $w = u + iv$ is called *locally sourceless* if

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

holds on G, and it is called *locally irrotational* provided

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

For a locally sourceless, locally irrotational flow w, the Cauchy–Riemann equations applied to the function $\overline{w} = u - iv$ are fulfilled, i.e., $\overline{w}$ is analytic on G. The *complex potential* f associated with such a flow w is any primitive on G of $\overline{w}$. If there is any at all, it is uniquely determined up to an additive complex constant. In general, a complex potential need not exist as a function. Most textbooks describe a complex potential as a multiple-valued analytic function, having single-valued analytic branches on every simply connected subdomain of G, and every branch having the same derivative $\overline{w}$. As an example, sources, sinks, and—more generally—spiral vortexes are given, leading to a complex potential of the form $f(z) = c \log z$, where $c \neq 0$ is a complex number. All worked examples exhibit complex potentials with logarithmic singularities only.

For further information we refer the reader to [4–7].

2. COMPLEX POTENTIALS

If we accept multiple-valued functions like the logarithm, we can prove the following result. It is surely well known to the workers in the field, but we could not trace a proof nor the statement in the literature.

THEOREM 2.1. *Let $G \subset \mathbb{C}$ be a $(n+1)$-connected domain, and let $D_1, D_2, \ldots, D_n$ denote the bounded connected components of $\mathbb{C} \setminus G$, that is the holes. Furthermore, let w be a locally sourceless, locally irrotational flow in G, and fix points $z_j \in D_j$. Then w has a complex potential f on G, and f has the form*

$$f(z) = g(z) + \sum_{j=1}^{n} c_j \log(z - z_j),$$

where g is a single-valued analytic function on G. Thereby, $c_1, \ldots, c_n$ are uniquely determined complex constants, independent of the particular choice of $z_1, \ldots, z_n$.

Proof. Choose closed integration paths $\gamma_k \subset G$ surrounding D_k once and none of the other holes, i.e., $n(\gamma_k, z_j) = \delta_{jk}$ for $j, k = 1, \ldots, n$. If f is a complex potential of the form above, we have for all $k = 1, \ldots, n$:

$$\oint_{\gamma_k} \overline{w(z)}\, dz = \oint_{\gamma_k} f'(z)\, dz = \oint_{\gamma_k} g'(z)\, dz + \sum_{j=1}^{n} c_j \oint_{\gamma_k} \frac{dz}{z - z_j}$$

$$= 0 + \sum_{j=1}^{n} c_j \cdot 2\pi i \delta_{jk} = 2\pi i c_k.$$

Hence, there is at most one tuple $(c_1, \ldots, c_n)$, such that the corresponding f is a complex potential for w. Moreover, this tuple is independent of the particular choice of $z_1, \ldots, z_n$.

On the other hand, defining $c_k := (1/2\pi i)\oint_{\gamma_k}\overline{w(z)}\, dz$ and the analytic function h on G by $h(z) := \overline{w(z)} - \sum_{j=1}^{n} c_j/(z - z_j)$, we conclude

$$\oint_{\gamma_k} h(z)\, dz = 2\pi i c_k - \sum_{j=1}^{n} c_j \cdot 2\pi i \delta_{jk} = 0 \qquad (k = 1, \ldots, n).$$

Cauchy's theorem for null-homologic cycles (see, e.g., [2, Theorem 5.7, p. 85]) now implies that $\oint_{\gamma} h(z)\, dz = 0$ holds for every closed integration path in G, i.e., h has a single-valued analytic primitive g on G. Defining f by $f(z) := g(z) + \sum_{j=1}^{n} c_j \log(z - z_j)$, we finally obtain

$$f'(z) = h(z) + \sum_{j=1}^{n} \frac{c_j}{z - z_j} = \overline{w(z)};$$

i.e., f is a complex potential for w. ∎

Remark. Assuming continuity of the locally sourceless, locally irrotational flow $w = u + iv$ up to the boundary of G, we can define g and f by integration not only on G, but on $\overline{G}$. If in addition ∂D_k is an analytic Jordan curve C_k, we obtain the following formula for the constant c_k, involving the circulation $\Gamma_k := \oint_{C_k} u\,dx + v\,dy$ along C_k and the flux $N_k := \oint_{C_k} u\,dy - v\,dx$ across C_k. With γ_k as in the proof of Theorem 2.1, we compute

$$c_k = \frac{1}{2\pi i}\oint_{\gamma_k} \overline{w(z)}\,dz = \frac{1}{2\pi i}\oint_{C_k} \overline{w(z)}\,dz = \frac{\Gamma_k + iN_k}{2\pi i}.$$

Here we actually used Cauchy's theorem and a limiting process to pass to the boundary.

The proof of Theorem 2.1 is similar to the proof of the more familiar theorem of Fisher concerning logarithmic singularities for harmonic functions (see [3, Theorem 1.3, p. 72]). We mention that Fisher's result for the real-valued harmonic function u can be obtained from our theorem by regarding the flow $w = u_x + iu_y$. However, we were unable to show that Fisher's result implies ours.

3. COMPLEX POTENTIALS IN INFINITELY CONNECTED DOMAINS

It is desirable to have a result like Theorem 2.1 for infinitely connected domains. The main obstacle is obviously the nonconvergence of the series of logarithm terms. However, in many cases, we can force convergence by adding polynomials, a well-known procedure in the proof of the Mittag–Leffler theorem. Again, the resulting complex potentials have logarithmic singularities only.

THEOREM 3.1. *Let D_k, $k \in \mathbb{N}$, denote a sequence of compact, connected, pair-wise disjoint subsets of $\mathbb{C}$ which only cluster at ∞, and consider the domain $G := \mathbb{C} \setminus \bigcup_{k=1}^{\infty} D_k$. Assume $0 \in G$. Furthermore, let w be a locally sourceless, locally irrotational flow on G and fix points $z_k \in D_k$. Then w has a complex potential f on G, and it has the form*

$$f(z) = g(z) + \sum_{k=1}^{\infty} c_k(\log(z - z_k) + p_k(z)),$$

where g is a single-valued analytic function on G, the sum is locally uniformly convergent in G, and p_k is a polynomial of the form

$$p_k(z) = -\log(-z_k) + \sum_{j=1}^{m_k} \frac{1}{j}\left(\frac{z}{z_k}\right)^j$$

with $m_k \in \mathbb{N}_0$. Thereby, the sequence (c_k) is uniquely determined and independent of the choice of (z_k).

With regard to the locally uniform convergence, fix a simply connected domain $U \subset G$ with $0 \in U$, and fix for every summand a single valued analytic branch $\log_k$ on $U - z_k$ of the logarithm. Then

$$\sum_{k=1}^{\infty} c_k \left(\log_k(z - z_k) - \log_k(-z_k) + \sum_{j=1}^{m_k} \frac{1}{j} \left(\frac{z}{z_k} \right)^j \right)$$

converges locally uniformly in U.

Proof. Using the assumption that the holes D_k only cluster at ∞, we can choose closed integration paths $\gamma_k \subset G$ surrounding D_k once and none of the other holes, i.e., $n(\gamma_k, z_j) = \delta_{jk}$ for $j, k \in \mathbb{N}$ as in the proof of Theorem 2.1. If f is a complex potential of the form above, we have for all $k \in \mathbb{N}$:

$$\begin{aligned} \oint_{\gamma_k} \overline{w(z)}\, dz &= \oint_{\gamma_k} f'(z)\, dz = \oint_{\gamma_k} g'(z)\, dz + \sum_{j=1}^{\infty} c_j \oint_{\gamma_k} \left(\frac{1}{z - z_j} + p_j'(z) \right) dz \\ &= 0 + \sum_{j=1}^{\infty} c_j \cdot 2\pi i \delta_{jk} = 2\pi i c_k. \end{aligned}$$

Hence, there is at most one sequence (c_k), such that the corresponding f is a complex potential for w. Moreover, this sequence is independent of the particular choice of (z_k).

On the other hand, define $c_k := (1/2\pi i)\oint_{\gamma_k} \overline{w(z)}\, dz$. Choose $m_k \in \mathbb{N}_0$ $(k \in \mathbb{N})$, such that

$$\left| c_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} \sum_{j=0}^{m_k - 1} \left(\frac{z}{z_k} \right)^j \right) \right| = \left| \frac{c_k}{z_k} \sum_{j=m_k}^{\infty} \left(\frac{z}{z_k} \right)^j \right| \le 2^{-k}$$

for $z \in G$, $|z| \le \frac{1}{2}|z_k|$. Because of $\lim_{k \to \infty} z_k = \infty$, it follows that

$$h(z) := \overline{w(z)} - \sum_{k=1}^{\infty} c_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} \sum_{j=0}^{m_k - 1} \left(\frac{z}{z_k} \right)^j \right)$$

converges locally uniformly in G. Hence,

$$\oint_{\gamma_k} h(z)\, dz = 2\pi i c_k - \sum_{\kappa=1}^{\infty} c_\kappa (2\pi i \delta_{k\kappa} + 0) = 0$$

for every $k \in \mathbb{N}$. Cauchy's theorem for null-homologic cycles (see, e.g., [2, Theorem 5.7, p. 85]) now implies that $\oint_\gamma h(z)\,dz = 0$ holds for every closed integration path in G; i.e., h has a single-valued analytic primitive g on G. Fix a simply connected domain $U \subset G$ and branches $\log_k$ as in the remark preceding the proof. Then

$$\int_0^z \sum_{k=1}^{\infty} c_k \left(\frac{1}{\zeta - z_k} + \frac{1}{z_k} \sum_{j=0}^{m_k - 1} \left(\frac{\zeta}{z_k} \right)^j \right) d\zeta$$
$$= \sum_{k=1}^{\infty} c_k \left(\log_k(z - z_k) - \log_k(-z_k) + \sum_{j=1}^{m_k} \frac{1}{j} \left(\frac{z}{z_k} \right)^j \right)$$

converges locally uniformly in U. Moreover,

$$\frac{d}{dz}\left(g(z) + \sum_{k=1}^{\infty} c_k \left(\log_k(z - z_k) - \log_k(-z_k) + \sum_{j=1}^{m_k} \frac{1}{j} \left(\frac{z}{z_k} \right)^j \right) \right)$$
$$= h(z) + \sum_{k=1}^{\infty} c_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} \sum_{j=0}^{m_k - 1} \left(\frac{z}{z_k} \right)^j \right) = \overline{w(z)}$$

for $z \in U$ as desired. ∎

As an example, we consider Kármán's vortex street: This street consists of two parallel rows of vortices. The distance between two consecutive vortices on the upper row is l, and every vortex has circulation $\Gamma > 0$. The vortices on the lower row all have circulation $-\Gamma$ and are located directly beneath the midpoint of two consecutive vortices of the upper row. The distance between the two rows is h. In our notation above, we may set $G := \mathbb{C} \setminus \{\pm z_n \colon n \in \mathbb{Z}\}$, where $z_n := l/4 + ih/2 + nl$. Let w be a locally sourceless, locally irrotational flow on G with the circulations above, i.e., $\oint_{|z \mp z_n| = \varepsilon} \overline{w(z)}\,dz = \pm\Gamma$ for every $n \in \mathbb{Z}$ and small positive ε. By Theorem 3.1, w has a complex potential f of the form

$$f(z) = g(z) + \sum_{n=-\infty}^{\infty} \frac{\Gamma}{2\pi i} (\log(z - z_n) + p_n(z))$$
$$- \sum_{n=-\infty}^{\infty} \frac{\Gamma}{2\pi i} (\log(z + z_n) + \tilde{p}_n(z)).$$

In our special case, the polynomials needed to enforce the convergence of f' are just constants:

$$f'(z) = g'(z) + \sum_{n=-\infty}^{\infty} \frac{\Gamma}{2\pi i}\left(\frac{1}{z - z_n} + \frac{1}{z_n}\right) - \sum_{n=-\infty}^{\infty} \frac{\Gamma}{2\pi i}\left(\frac{1}{z + z_n} - \frac{1}{z_n}\right).$$

Note that both sums are converging absolutely (but would be diverging without the added constants). Summing symmetrically yields

$$\begin{aligned}
&\frac{\Gamma}{2\pi i}\sum_{n=-\infty}^{\infty}\left(\frac{1}{z - z_0 - nl} + \frac{1}{z_0 + nl}\right) \\
&\quad - \frac{\Gamma}{2\pi i}\sum_{n=-\infty}^{\infty}\left(\frac{1}{z + z_0 + nl} - \frac{1}{z_0 + nl}\right) \\
&= \frac{\Gamma}{2\pi i}\left(\frac{1}{z - z_0} + \frac{1}{z_0} + \sum_{n=1}^{\infty}\left(\frac{2(z - z_0)}{(z - z_0)^2 - n^2 l^2} + \frac{2z_0}{z_0^2 - n^2 l^2}\right)\right) \\
&\quad - \frac{\Gamma}{2\pi i}\left(\frac{1}{z + z_0} - \frac{1}{z_0} + \sum_{n=1}^{\infty}\left(\frac{2(z + z_0)}{(z + z_0)^2 - n^2 l^2} - \frac{2z_0}{z_0^2 - n^2 l^2}\right)\right) \\
&= \frac{\Gamma}{2\pi i}\frac{\pi}{l}\left(\cot\frac{\pi(z - z_0)}{l} - \cot\frac{\pi(z + z_0)}{l} + 2\cot\frac{\pi z_0}{l}\right),
\end{aligned}$$

i.e.,

$$f(z) = \tilde{g}(z) + \frac{\Gamma}{2\pi i}\log\frac{\sin\dfrac{\pi(z - z_0)}{l}}{\sin\dfrac{\pi(z + z_0)}{l}},$$

where $\tilde{g}(z) := g(z) + 2z\cot(\pi z_0/l)$ defines a single-valued analytic function on G. On the other hand, every complex potential of this form yields a flow as desired.

4. STREAMLINES AND STAGNATION POINTS

A streamline is the trajectory of a fluid particle in the flow $\mathbf{w} = (u, v)^t$. If $\mathbf{x}(t) = (x(t), y(t))^t$ denotes the position of the fluid particle at time t, this means $\dot{\mathbf{x}} = \mathbf{w}(\mathbf{x})$; hence $u(\mathbf{x}) \cdot \dot{y} = v(\mathbf{x}) \cdot \dot{x}$. For a locally sourceless, locally irrotational flow $w = u + iv$ with complex potential f, we calculate

$$\begin{aligned}
&\operatorname{Im} f(z(t_2)) - \operatorname{Im} f(z(t_1)) \\
&\quad = \operatorname{Im} \int_{t_1}^{t_2} f'(z(t))\dot{z}(t)\,dt \\
&\quad = \operatorname{Im} \int_{t_1}^{t_2} (u(z(t)) - iv(z(t)))(\dot{x}(t) + i\dot{y}(t))\,dt \\
&\quad = \int_{t_1}^{t_2} (u(z(t))\dot{y}(t) - v(z(t))\dot{x}(t))\,dt = 0; \qquad (1)
\end{aligned}$$

i.e., $\operatorname{Im} f$ is constant on every trajectory. If f is not single valued, we actually have to take the analytic continuation along the trajectory.

On the other hand, let $\operatorname{Im} f$ be constant on an arbitrary differentiable curve C: $s \mapsto (\xi(s), \eta(s))^t$. By (1), this implies $u(\zeta(s))\eta'(s) = v(\zeta(s))\xi'(s)$ for every s; i.e., the tangent vector on C is parallel to w at every point of C.

Usually, $\operatorname{Im} f = \mathit{const.}$ is said to be the defining condition for streamlines. But one should be more careful. We distinguish two cases:

1. Suppose $s_1 < s_2$ and $f'(\zeta(s)) \neq 0$, for every $s \in [s_1, s_2]$. First, this implies that $\operatorname{Im} f = \operatorname{Im} f(\zeta(s_0))$ implicitly defines a smooth curve in a neighborhood of $\zeta(s_0)$ for every $s_0 \in [s_1, s_2]$. Therefore, we may assume $\zeta'(s) \neq 0$ for every $s \in [s_1, s_2]$. Especially the tangent vector of C always points in the same (a) or always points in the opposite (b) direction as $\mathbf{w}$. Because of $\min_{s \in [s_1, s_2]} |f'(s)| > 0$, we conclude that a fluid particle in $\zeta(s_1)$ reaches $\zeta(s_2)$ in finite time (case a), respectively, a fluid particle in $\zeta(s_2)$ reaches $\zeta(s_1)$ in finite time (case b).

2. $f'(z_0) = 0$. Then $f = f(z_0) + g^k$ in a neighborhood U of z_0, where $k \in \mathbb{N}$, $k > 1$, and g is conformal with $g(z_0) = 0$. Hence, $\{z \in U$: $\operatorname{Im} f(z) = \operatorname{Im} f(z_0)\} = \sqrt[k]{g^{-1}(\mathbb{R})}$ consists of $2k$ smooth curves starting in z_0, whereby the angle between two consecutive curves is π/k. Let C: $[0, s_1] \to \mathbb{C}$, $s \mapsto \zeta(s)$ be one of these curves, where $\zeta(0) = z_0$ and s denotes arclength. Especially this implies $|\zeta(s) - z_0| \leq s$ for every $s \in [0, s_1]$. Since f' has a zero of order $k - 1$ in z_0, there is a constant $c < \infty$, such that $|f'(z)| \leq c|z - z_0|^{k-1}$ for all z near z_0. For $s_0 \in (0, s_1]$, we therefore have

$$\int_0^{s_0} \frac{ds}{|w(\zeta(s))|} \geq \frac{1}{c} \int_0^{s_0} \frac{ds}{|\zeta(s) - z_0|^{k-1}} \geq \frac{1}{c} \int_0^{s_0} \frac{ds}{|s|^{k-1}} = \infty.$$

Since ds is the arclength element and since $|w(z)|$ is the velocity of a fluid particle at the point z, the first integral is the time that a fluid particle needs in the flow w to move along C between $\zeta(s_0)$ and z_0; i.e., z_0 cannot be reached in finite time. Such a point is called a *stagnation point*.

The considerations above show that the trajectories of fluid particles in the flow w are given by $\operatorname{Im} f(z) = const.$ *and* $f'(z) \neq 0$.

5. FLOWS AROUND CONTOURS

In this section, we apply the previous results to the study of flows around finitely many contours $D_1, \dots, D_n$, where every D_j is assumed to be a bounded domain in $\mathbb{C}$ with an analytic Jordan curve C_j as boundary. A flow w on $G := \mathbb{C} \setminus (D_1 \cup \cdots \cup D_n)$ is called a flow *around* D_j (or *around* C_j) if w is tangential to C_j in every point of C_j. This is the case if and only if the flux of w across γ, i.e., $\int_\gamma \mathbf{w} \cdot \mathbf{n}\, ds$, vanishes for every subarc γ of C_j.

THEOREM 5.1. *Let $G \subset \mathbb{C}$ denote an unbounded domain, with n pairwise disjoint analytic Jordan curves $C_1, \dots, C_n$ as boundary. Fix z_j in the interior of C_j. Assume a locally sourceless, locally irrotational flow w around $C_1, \dots, C_n$, continuous on the closure $\overline{G} = G \cup C_1 \cup \cdots \cup C_n$, such that $w_\infty := \lim_{z \to \infty} w(z)$ exists. Then the complex potential f associated with w has the form*

$$f(z) = g(z) + \overline{w_\infty} z + \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi i} \log(z - z_j),$$

where $\Gamma_j := \oint_{C_j} \overline{w(z)}\, dz \in \mathbb{R}$ and g is single-valued analytic on G with finite limit $\lim_{z \to \infty} g(z)$. Moreover, the real potential function $\psi := \operatorname{Im} f$ is single-valued on $\overline{G}$, and $\operatorname{Im} f$ is constant on C_j for every $j = 1, \dots, n$.

Proof. Using Theorem 2.1, we have

$$f(z) = \tilde{g}(z) + \sum_{j=1}^{n} c_j \log(z - z_j),$$

where $\tilde{g}$ is a single-valued analytic function on G. For sufficiently large R, we represent $\tilde{g}$ by a Laurent series:

$$\tilde{g}(z) = \sum_{j=-\infty}^{\infty} a_j z^j, \qquad \tilde{g}'(z) = \sum_{j=-\infty}^{\infty} j a_j z^{j-1} \qquad (|z| > R).$$

The existence of the finite limit $w_\infty = \lim_{z \to \infty} \overline{f'(z)} = \lim_{z \to \infty} \overline{\tilde{g}'(z)}$ forces

$$\tilde{g}'(z) = \overline{w_\infty} + \sum_{j=-\infty}^{0} j a_j z^{j-1} \quad \text{and} \quad \tilde{g}(z) = \overline{w_\infty} z + \sum_{j=-\infty}^{0} a_j z.$$

Hence we conclude

$$f(z) = g(z) + \overline{w_\infty} z + \sum_{j=1}^{n} c_j \log(z - z_j),$$

where g is single-valued analytic on G with finite limit a_0 in ∞.

By the remark following Theorem 2.1, f and g are defined on $\overline{G}$, and we have $c_j = \oint_{C_j} \overline{w(z)}\, dz = (\Gamma_j + iN_j)/2\pi i$, where Γ_j, respectively N_j, denotes the circulation along, respectively, flux across, C_j. Since w is a flow around C_j, every flux N_j vanishes; i.e., $c_j = \Gamma_j / 2\pi i$ is real for every $j = 1, \dots, n$. In particular,

$$\psi(z) = \operatorname{Im} f(z) = \operatorname{Im}\left(g(z) + \overline{w_\infty} z\right) - \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi} \log|z - z_j|;$$

hence ψ is single-valued on $\overline{G}$. Therefore,

$$\psi(z_2) - \psi(z_1) = \operatorname{Im}\left(f(z_2) - f(z_1)\right) = \operatorname{Im} \int_{z_1}^{z_2} f'(z)\, dz = \operatorname{Im} \int_{z_1}^{z_2} \overline{w(z)}\, dz$$

holds not only locally for a fixed branch of f, but throughout $\overline{G}$. Fix $z_1, z_2 \in C_j$ and let γ denote the subarc of C_j from z_1 to z_2. Then

$$\psi(z_2) - \psi(z_1) = \operatorname{Im} \int_\gamma \overline{w(z)}\, dz = \int_\gamma \mathbf{w} \cdot \mathbf{n}\, ds = 0.$$

Hence ψ is constant on C_j for every $j = 1, \dots, n$. ∎

We remark that ψ has a harmonic continuation to a neighborhood of C_j for every $j = 1, \dots, n$, since C_j is an analytic Jordan curve and ψ is constant on C_j. By the same reason, f has an analytic continuation, too. More precisely, suppose a subdomain $U \subseteq G$ and a subarc $\gamma \subseteq C_j$ with $\gamma \subseteq \partial U$, such that there is a single-valued branch of the complex potential in U. Then this branch has an analytic continuation beyond γ.

The formulas from Theorem 5.1 can be used to prove the existence of such flows. It is of utmost advantage to know the necessary form of the associated complex potential.

THEOREM 5.2. *Let $G \subset \mathbb{C}$ denote an unbounded domain with n pairwise disjoint analytic Jordan curves $C_1, \dots, C_n$ as boundary. Given $\Gamma_1, \dots, \Gamma_n \in \mathbb{R}$ and $w_\infty \in \mathbb{C}$, there exists exactly one locally sourceless, locally irrotational flow w around $C_1, \dots, C_n$, continuous on the closure $\overline{G} = G \cup C_1 \cup \dots \cup C_n$, satisfying $\oint_{C_j} \overline{w(z)}\, dz = \Gamma_j$ and $\lim_{z \to \infty} w(z) = w_\infty$.*

Proof. Fix z_j in the interior of C_j. By Theorem 5.1, every flow with the desired properties has exactly one complex potential f of the form

$$f(z) = g(z) + \overline{w_\infty} z + \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi i} \log(z - z_j),$$

where

- $\operatorname{Im} f = 0$ on C_1 and $\operatorname{Im} f = \mathit{const.}$ on C_j for every $j = 2, \dots, n$,
- g is single-valued analytic on $\hat{G} := G \cup \{\infty\}$ with $\operatorname{Re} g(\infty) = 0$.

(Recall that the complex potential is unique up to an additive complex constant.) On the other hand, for every f of the form above, $w := \overline{f'}$ is a flow as desired. So we are done once we can prove that there is exactly one such f.

Note that the Dirichlet problem in $\hat{G}$ is solvable. Therefore there exists a (bounded) harmonic function h on $\hat{G}$, having the boundary values

$$\begin{aligned} h(z) &= -\operatorname{Im}\left(\overline{w_\infty} z + \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi i} \log(z - z_j) \right) \\ &= \operatorname{Im}(w_\infty \bar{z}) + \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi} \log|z - z_j| \end{aligned}$$

for all $z \in C_1 \cup \cdots \cup C_n$. Moreover, let ω_j denote the harmonic measure for C_j; i.e., ω_j is harmonic in $\hat{G}$ satisfying $\omega_j = \delta_{jk}$ on C_k for all $j, k = 1, \dots, n$. If f is of the form above with $\operatorname{Im} f = c_j$ ($j = 2, \dots, n$), then

$$h + \sum_{j=2}^{n} c_j \omega_j = \operatorname{Im} g,$$

since the boundary values are equal. On the other side, assume that there are $c_2, \dots, c_n$, such that $h + \sum_{j=2}^{n} c_j \omega_j$ is the imaginary part of a single-valued analytic function on $\hat{G}$. Then there is exactly one single-valued analytic function g on $\hat{G}$ with $\operatorname{Re} g(\infty) = 0$ and $\operatorname{Im} g = h + \sum_{j=2}^{n} c_j \omega_j$. Defining

$$f(z) = g(z) + \overline{w_\infty} z + \sum_{j=1}^{n} \frac{\Gamma_j}{2\pi i} \log(z - z_j)$$

yields a complex potential f with the properties above. Looking at the boundary values of $\operatorname{Im} f$, we observe that different tuples $(c_2, \dots, c_n)$ lead to different functions f. So we are done once we can prove that there is exactly one tuple $(c_2, \dots, c_n)$ such that $h + \sum_{j=2}^{n} c_j \omega_j$ is the imaginary part of a single-valued analytic function on $\hat{G}$. This, however, follows immediately from the well-known invertibility of the period matrix associated with h (see [1, Lemma 2.4] or [3, Theorems 1.2 and 1.7, p. 72 ff.]). (Using the references, it is necessary to invert the domain $\hat{G}$; say we use the Moebius transform T given by $T(z) := 1/(z - z_1)$. The result is a bounded domain in $\mathbb{C}$. Since the harmonic measures change just by composition with this inversion, we are done.) ∎

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